MATH20132 Calculus of Several Variables. 2020-21

Solutions to Problems 1 : Limits

Questions 1 - 4 concern the limits of functions. *The* ε - δ *definition of a limit is that* $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *has the limit* $\mathbf{b} \in \mathbb{R}^m$ *at* $\mathbf{a} \in U$ *iff*

$$\forall \varepsilon > 0, \exists\, \delta > 0 : \forall \mathbf{x} \in \mathbb{R}^n,\ 0 < |\mathbf{x} - \mathbf{a}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{b}| < \varepsilon.$$

1. By verifying the ε - δ definition of limit show that the scalar-valued function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y)^T \mapsto x + y$ has limit 5 at $\mathbf{a} = (2, 3)^T$.

Hint At some point in verifying the definition you assume $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$. In particular this gives **two** pieces of information, namely that $|x - 2| < \delta$ and $|y - 3| < \delta$.

Solution Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/2$. Assume $\mathbf{x} = (x,\ y)^T$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$ in which case the coordinates of $\mathbf{x}$ satisfy

$$|x - 2| < \delta \quad \text{and} \quad |y - 3| < \delta. \tag{1}$$

Then, with $\mathbf{f}(\mathbf{x}) = x + y$ and $\mathbf{b} = 5$,

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{b}| &= |(x + y) - 5| = |(x - 2) + (y - 3)| \le |x - 2| + |y - 3| \\ &\qquad \text{by the triangle inequality} \\ &< 2\delta \qquad \text{by (1)} \\ &= 2\,(\varepsilon/2) = \varepsilon. \end{aligned}$$

Hence we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$, i.e. $\lim_{\mathbf{x}\to\mathbf{a}} (x + y) = 5$.

2. By verifying the ε - δ definition of limit show that the scalar-valued function $g : \mathbb{R}^2 \to \mathbb{R},\ (x, y)^T \mapsto xy + x + y$ has limit 11 at $\mathbf{a} = (2, 3)^T$.

Hint Perhaps start by proving that

$$xy + x + y - 11 = (x - 2)\,(y - 3) + 4\,(x - 2) + 3\,(y - 3)\,.$$

Deduce that if $|x - 2| < \delta$, $|y - 3| < \delta$ and $\delta \le 1$ then $|xy - 6| < 8\delta$. Now look at the definition of limit.

Solution Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/6)$. Assume $\mathbf{x}$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$ so

$$|x-2| < \delta \quad \text{and} \quad |y-3| < \delta. \tag{2}$$

Rewrite $xy + x + y - 11$ in terms of $x - 2$ and $y - 3$:

$$\begin{aligned} xy + x + y - 11 &= (x-2)(y-3) + 4x + 3y - 17 \\ &= (x-2)(y-3) + 4(x-2) + 3(y-3). \end{aligned}$$

The triangle inequality gives

$$\begin{aligned} |xy + x + y - 11| &\leq |x-2||y-3| + 4|x-2| + 3|y-3| \\ &< \delta^2 + 4\delta + 3\delta, \end{aligned} \tag{3}$$

by (2). The δ^2 factor is unnecessarily complicated yet we are assuming $\delta = \min(1, \varepsilon/6) \leq 1$ which implies $\delta^2 \leq \delta$ in which case (3) becomes

$$|xy + x + y - 11| < \delta + 4\delta + 3\delta = 8\delta.$$

Since also $\delta = \min(1, \varepsilon/8) \leq \varepsilon/8$ we have

$$|xy + x + y - 11| < 8\delta \leq 8\left(\frac{\varepsilon}{8}\right) = \varepsilon.$$

Hence we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} (xy + x + y) = 6$.

3. By verifying the ε - δ definition of limit show that the vector-valued function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x + y \\ x - 3y \end{pmatrix},$$

has limit $(7, -7)^T$ at $\mathbf{a} = (2, 3)^T$.

Note For practice I have asked you to verify the definition, **not** to use any result from the course that would allow you to look at each component separately.

Solution Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/\sqrt{17}$. Assume $\mathbf{x} = (x, y)^T$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$ so, again, $|x - 2| < \delta$ and $|y - 3| < \delta$. With

$\mathbf{b} = (7, -7)^T$ consider

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{b}|^2 &= \left| \begin{pmatrix} 2x + y \\ x - 3y \end{pmatrix} - \begin{pmatrix} 7 \\ -7 \end{pmatrix} \right|^2 \\ &= \left| \begin{pmatrix} 2(x-2) + (y-3) \\ (x-2) - 3(y-3) \end{pmatrix} \right|^2 . \end{aligned}$$

I have written this in terms of $x - 2$ and $y - 3$ since I know I can make them small. Continue, using the definition of $|...|$ on $\mathbb{R}^2$,

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{b}|^2 &= \big(2(x-2) + (y-3)\big)^2 + \big((x-2) - 3(y-3)\big)^2 \\ &= 4(x-2)^2 + 4(x-2)(y-3) + (y-3)^2 \\ &\qquad + (x-2)^2 - 6(x-2)(y-3) + 9(y-3)^2 \\ &= 5(x-2)^2 - 2(x-2)(y-3) + 10(y-3)^2 . \end{aligned}$$

The negative sign on the middle term is a possible problem when applying upper bounds for $|x - 2|$ and $|y - 3|$. We remove this by using the triangle inequality:

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{b}|^2 &= \left|5(x-2)^2 - 2(x-2)(y-3) + 10(y-3)^2\right| \\ &\le 5(x-2)^2 + 2|x-2||y-3| + 10(y-3)^2 . \end{aligned}$$

Thus

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}|^2 < 5\delta^2 + 2\delta^2 + 10\delta^2 = 17\delta^2.$$

Taking roots gives

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}| < \sqrt{17}\delta = \sqrt{17}\left(\frac{\varepsilon}{\sqrt{17}}\right) = \varepsilon.$$

Hence $\mathbf{f}$ has limit $(7, -7)^T$ at $(2, 3)^T$.

Alternative solution Given $\mathbf{x} \in \mathbb{R}^n$, the triangle inequality gives $|\mathbf{x}| \le \sum_{i=1}^n |x^i|$. Used above with $n = 2$ this gives

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{b}| &\le |2(x-2) + (y-3)| + |(x-2) - 3(y-3)| \\ &\le 2|x-2| + |y-3| + |x-2| + 3|y-3| , \end{aligned}$$

by further applications of the triangle inequality. Thus

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}| \leq 3|x-2| + 4|y-3|.$$

The choice of $\delta = \varepsilon/7$ will now suffice.

4. By verifying the ε - δ definition of limit show that the vector-valued $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ xy+x+y \end{pmatrix},$$

has limit $(5, 11)^T$ at $\mathbf{a} = (2, 3)^T$.

Hint try to make use of the results used in Questions 1 and 2.

Solution Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/\sqrt{68} > 0$. Assume $0 < |\mathbf{x} - \mathbf{a}| < \delta$. Let $\mathbf{b} = (5, 11)^T$. Then

$$\begin{aligned} |\mathbf{h}(\mathbf{x}) - \mathbf{b}|^2 &= \left| \begin{pmatrix} x+y-5 \\ xy+x+y-11 \end{pmatrix} \right|^2 \\ &= (x+y-5)^2 + (xy+x+y-11)^2. \end{aligned}$$

Do NOT multiply this out, Instead, use the results of Question 2, namely if $|x-2| < \delta$, $|y-3| < \delta$ and $\delta \leq 1$ then $|xy+x+y-11| < 8\delta$. Under the same conditions,

$$|x+y-5| = |(x-2)+(y-3)| \leq |x-2| + |y-3| \leq 2\delta.$$

Thus

$$|\mathbf{h}(\mathbf{x}) - \mathbf{b}|^2 \leq (2\delta)^2 + (8\delta)^2 = 68\delta^2.$$

For such $\mathbf{x}$ we have

$$|\mathbf{h}(\mathbf{x}) - \mathbf{b}|^2 \leq 68\delta^2 = 68\left(\varepsilon/\sqrt{68}\right)^2 = \varepsilon^2.$$

Hence $|\mathbf{h}(\mathbf{x}) - \mathbf{b}| < \varepsilon$ and we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{h}(\mathbf{x}) = \mathbf{b}$.

5. Assume $f, g : D \subseteq \mathbb{R}^n \to \mathbb{R}$ are **scalar-valued** functions with domain D containing a deleted neighbourhood of $\mathbf{a} \in \mathbb{R}^n$. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = b \in \mathbb{R}$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = c \in \mathbb{R}$ prove that

i. $\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x}) + g(\mathbf{x})) = b + c$,

ii. $\lim_{\mathbf{x}\to\mathbf{a}} \left(f(\mathbf{x})\, g(\mathbf{x}) \right) = bc$ and

iii. $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) / g(\mathbf{x}) = b/c$ provided $c \neq 0$.

Hint No new ideas are required, the proofs are identical to those for functions of one variable.

Solution Let $\varepsilon > 0$ be given.

i. $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = b$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = c$ imply

$$\exists \delta_1 > 0 : \forall \mathbf{x} : 0 < |\mathbf{x} - \mathbf{a}| < \delta_1 \Longrightarrow |f(\mathbf{x}) - b| < \varepsilon/2,$$
$$\exists \delta_2 > 0 : \forall \mathbf{x} : 0 < |\mathbf{x} - \mathbf{a}| < \delta_2 \Longrightarrow |g(\mathbf{x}) - c| < \varepsilon/2.$$

Choose $\delta = \min(\delta_1, \delta_2)$ and assume $\mathbf{x} : 0 < |\mathbf{x} - \mathbf{a}| < \delta$. For such $\mathbf{x}$ we have, starting with the triangle inequality,

$$|(f(\mathbf{x}) + g(\mathbf{x})) - (b + c)| \leq |f(\mathbf{x}) - b| + |g(\mathbf{x}) - c| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus we have verified the ε-δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \left(f(\mathbf{x}) + g(\mathbf{x}) \right) = b + c$.

ii. Left to student but starts with the identity

$$f(\mathbf{x})\, g(\mathbf{x}) - bc = (f(\mathbf{x}) - b)\, g(\mathbf{x}) + (g(\mathbf{x}) - c)\, b.$$

You will need to prove that $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = c$ implies there exists $\delta > 0$ such that if $\mathbf{x}$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$ then $|g(\mathbf{x})| \leq 1 + |c|$.

iii. Left to student but starts by proving $\lim_{\mathbf{x}\to\mathbf{a}} 1/g(\mathbf{x}) = 1/c$. This, in turn, starts from

$$\frac{1}{g(\mathbf{x})} - \frac{1}{c} = -\frac{g(\mathbf{x}) - c}{g(\mathbf{x})\, c}.$$

You will need to prove that $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = c$ and $c \neq 0$ implies there exists $\delta > 0$ such that if $\mathbf{x}$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$ then $|g(\mathbf{x})| \geq |c|/2$.

The following is a corollary of Question 5.

6. Assume $\mathbf{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are **vector-valued** functions with domain D containing a deleted neighbourhood of $\mathbf{a} \in \mathbb{R}^n$. If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \in \mathbb{R}^m$ and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{c} \in \mathbb{R}^m$ prove that

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) \bullet \mathbf{g}(\mathbf{x}) = \mathbf{b} \bullet \mathbf{c}.$$

Here $\bullet$ is the scalar or dot product of vectors.

Hint Make use of the previous question.

Solution From the definition of the scalar product is

$$\mathbf{f}(\mathbf{x}) \bullet \mathbf{g}(\mathbf{x}) = \sum_{i=1}^{m} f^i(\mathbf{x})\, g^i(\mathbf{x})\,.$$

This is a sum of products of scalar-valued functions and so the result follows from the results on limits of scalar-valued functions in Question 5.

7. **Lemma from Lecture Notes, limits along straight lines** *Assume* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is a vector-valued function with domain* A *containing a deleted neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$*. Assume* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$*. Then, for any non-zero vector* $\mathbf{v} \in \mathbb{R}^n$*, the directional limit of* $\mathbf{f}$ *at* $\mathbf{a}$ *from the direction* $\mathbf{v}$ *exists and further*

$$\lim_{t\to 0+} \mathbf{f}(\mathbf{a}+t\mathbf{v}) = \mathbf{b}.$$

Prove this.

Hint This is a particular form of the Composite Rule for limits and so we can follow the outline of all proofs of such results.
Start by considering the ε - δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ and
finish by verifying the ε - δ definition of $\lim_{t\to 0+} \mathbf{f}(\mathbf{a}+t\mathbf{v}) = \mathbf{b}$.

Solution Let $\varepsilon > 0$ be given. Then by the definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ there exists $\delta > 0$ such that

$$0 < |\mathbf{x}-\mathbf{a}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{b}| < \varepsilon. \tag{4}$$

Assume $0 < t < \delta/|\mathbf{v}|$ then $\mathbf{a}+t\mathbf{v}$ satisfies

$$0 < |(\mathbf{a}+t\mathbf{v}) - \mathbf{a}| = |t|\,|\mathbf{v}| < \frac{\delta}{|\mathbf{v}|}\,|\mathbf{v}| = \delta.$$

Thus, by (4) with $\mathbf{x} = \mathbf{a} + t\mathbf{v}$, we deduce $|\mathbf{f}(\mathbf{a}+t\mathbf{v}) - \mathbf{b}| < \varepsilon$. That is

$$0 < t < \delta/|\mathbf{v}| \implies |\mathbf{f}(\mathbf{a}+t\mathbf{v}) - \mathbf{b}| < \varepsilon.$$

Hence we have verified the definition of $\lim_{x\to 0+} \mathbf{f}(\mathbf{a}+t\mathbf{v}) = \mathbf{b}$.

The result of the previous question can be written symbolically as

$$\boxed{\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \implies \forall\, \mathbf{v},\ \lim_{t\to 0+} \mathbf{f}(\mathbf{a}+t\mathbf{v}) = \mathbf{b}.} \tag{5}$$

The contrapositive can be used to prove limits do not exist.

8. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{for } \mathbf{x} = (x, y)^T \neq \mathbf{0} \quad \text{and} \quad f(\mathbf{0}) = 0.$$

i. Find $\lim_{t\to 0+} f(t\mathbf{e}_1)$ and $\lim_{t\to 0+} f(t\mathbf{e}_2)$ where $\mathbf{e}_1 = (1,0)^T$ and $\mathbf{e}_2 = (0,1)^T$ are the two standard basis vectors for $\mathbb{R}^2$.

ii. Prove that f does not have a limit at $\mathbf{0}$.

Solution i For $t \neq 0$ we have

$$f(t\mathbf{e}_1) = \frac{t^2 - 0^2}{t^2 + 0^2} = 1 \quad \text{so} \quad \lim_{t\to 0+} f(t\mathbf{e}_1) = 1.$$

Similarly, for $t \neq 0$ we have

$$f(t\mathbf{e}_2) = \frac{0^2 - t^2}{0^2 + t^2} = -1 \quad \text{so} \quad \lim_{t\to 0+} f(t\mathbf{e}_1) = -1.$$

ii. The implication in (5) is that if the limit exists then all the directional derivatives exist and are the same. We have found two different directional limits therefore the limit of f at $\mathbf{0}$ cannot exist.

9. **Lemma from lecture notes, limits along curves**. *Assume* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *where* A *contains a deleted neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$. *Assume* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ *exists. Assume* $\mathbf{g} : (0, \eta) \to A \setminus \{\mathbf{a}\}$ *with* $\lim_{t\to 0+} \mathbf{g}(t) = \mathbf{a}$. *Then*

$$\lim_{t\to 0+} \mathbf{f}(\mathbf{g}(t)) = \mathbf{b}.$$

Prove this.

Hint Question 7 is a special case of this result, so use the same method of proof. **Start** by looking at the ε - δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$ and **finish** verifying the ε - δ definition of $\lim_{t\to 0+} \mathbf{f}(\mathbf{g}(t))$.

Solution Let $\varepsilon > 0$ be given. From the definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ there exists $\delta_1 > 0$ such that

$$0 < |\mathbf{x} - \mathbf{a}| < \delta_1 \implies |\mathbf{f}(\mathbf{x}) - \mathbf{b}| < \varepsilon. \tag{6}$$

Choose $\varepsilon = \delta_1$ in the definition of $\lim_{t\to 0+} \mathbf{g}(t) = \mathbf{a}$ to find $\delta_2 > 0$ such that if

$$0 < t < \delta_2 \implies |\mathbf{g}(t) - \mathbf{a}| < \delta_1.$$

We can assume $\delta_2 \leq \eta$ (for if $\delta_2 > \eta$ then replace δ_2 by η).

We also have the assumption that $\mathbf{g} : (0, \eta) \to A \setminus \{\mathbf{a}\}$ so, in particular, $\mathbf{g}(t) \neq \mathbf{a}$ for $0 < t < \delta_2$. Hence

$$0 < t < \delta_2 \implies 0 < |\mathbf{g}(t) - \mathbf{a}| < \delta_1. \tag{7}$$

Combine (7) and (6) with $\mathbf{x} = \mathbf{g}(t)$, to get

$$0 < t < \delta_2 \implies 0 < |\mathbf{g}(t) - \mathbf{a}| < \delta_1 \implies |\mathbf{f}(\mathbf{g}(t)) - \mathbf{b}| < \varepsilon.$$

Hence we have verified the definition of $\lim_{t\to 0+} \mathbf{f}(\mathbf{g}(t)) = \mathbf{b}$.

10. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{\left(x^2 - y\right)^2}{x^4 + y^2} \quad \text{for } \mathbf{x} = (x, y)^T \neq \mathbf{0} \quad \text{and} \quad f(\mathbf{0}) = 1.$$

i. Prove that $\lim_{t\to 0+} f(t\mathbf{v}) = 1$ for every non-zero vector $\mathbf{v}$.

   Hint: write $\mathbf{v} = (h, k)^T$ in order to get an expression for $f(t\mathbf{v})$. Be careful when $k = 0$.

ii. By considering the limit along the curve that is the image of $\mathbf{g}(t) = \left(t, t^2\right)^T$, prove that the function f does **not** have a limit at 0.

Hint Use the result of Question 9.

This is an example promised in the notes, of a function where the directional limit exists and are equal for all directions but the limit does not exist. That is

$$\forall \mathbf{v},\ \lim_{t\to 0+} \mathbf{f}(\mathbf{a} + t\mathbf{v}) = \mathbf{b} \ \not\Longrightarrow\ \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}.$$

That is, the converse of (5) *is false.*

Solution i. Follow the hint and write $\mathbf{v} = (h, k)^T$. There are two cases, $k \neq 0$ and $k = 0$. In the **first case**, $k \neq 0$, we find, for $t \neq 0$, that.

$$f(t\mathbf{v}) = \frac{\left((th)^2 - tk\right)^2}{(th)^4 + (tk)^2} = \frac{\left(th^2 - k\right)^2}{t^2h^4 + k^2}.$$

Then $f(t\mathbf{v}) \to (-k)^2 / k^2 = 1$ as $t \to 0+$. In the **second case**, when $k = 0$, then for $t \neq 0$,

$$f(t\mathbf{v}) = \frac{\left((th)^2\right)^2}{(th)^4} = 1$$

so, again, $f(t\mathbf{v}) \to 1$ as $t \to 0+$.

Hence, for all unit $\mathbf{v}$ we have $\lim_{t\to 0+} f(t\mathbf{v}) = 1$, i.e. all direction limits equal 1.

ii. For $t \neq 0$ (when also $\mathbf{g}(t) \neq \mathbf{0}$),

$$f(\mathbf{g}(t)) = \frac{(t^2 - t^2)^2}{t^2 + t^2} = 0,$$

so $f(\mathbf{g}(t)) \to 0$ as $t \to 0+$.

Assume f has a limit at $\mathbf{0}$. Then by the Lemma quoted in Question 9, f will have the same limit along all curves. Yet the limits along the two curves $(0,t)^T$, $t > 0$ and $(t,t^2)^T$, $t > 0$ are different. This contradiction means that f has no limit at $\mathbf{0}$.

The following is a particularly important question. Make sure you attempt all parts which illustrate points made in the lectures and are used in later questions.

11. Find the following limits if they exist:

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^2 + y^2}$; (ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$;

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$; (iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$;

(v) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4 + y^2}$; (vi) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}$.

Hint First try to show they have no limit by finding different directions (and even different curves) along which the function has different limits. If you cannot find any counterexamples try to prove the limit exists, normally by applying the Sandwich Rule.

Solution i. If $\mathbf{x} = (x,y)^T \in \mathbb{R}^2$ then $|\mathbf{x}|^2 = x^2 + y^2$ and $|x|, |y| \leq |\mathbf{x}|$. Therefore, for $\mathbf{x} \neq \mathbf{0}$,

$$|f(\mathbf{x})| \leq \frac{|x|^4 + |y|^4}{|x^2 + y^2|} \leq \frac{|\mathbf{x}|^4 + |\mathbf{x}|^4}{|\mathbf{x}|^2} = 2|\mathbf{x}|^2 .$$

Then, by the Sandwich Rule, $f(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{0}$. Hence the limit exists and is 0.

ii. Let

$$f(\mathbf{x}) = \frac{xy}{x^2 + y^2}$$

for $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$, $\mathbf{x} \neq \mathbf{0}$. If $\mathbf{v}_1 = (1,1)^T / \sqrt{2}$ then

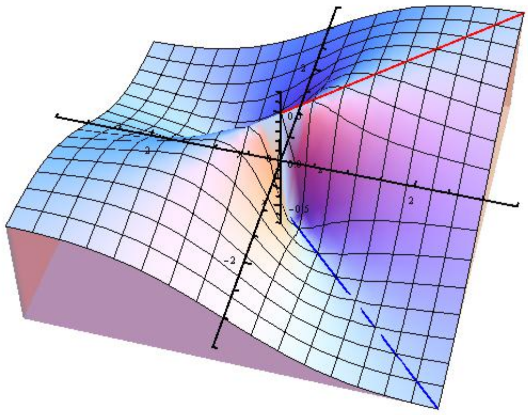
$$f(t\mathbf{v}_1) = \frac{t^2}{t^2 + t^2} = \frac{1}{2},$$

for all $t \neq 0$, so $\lim_{t \to 0+} f(t\mathbf{v}_1) = 1/2$. Yet if $\mathbf{v}_2 = (1, -1)^T / \sqrt{2}$ then

$$f(t\mathbf{v}_2) = \frac{-t^2}{t^2 + t^2} = -\frac{1}{2},$$

for all $t \neq 0$, so $\lim_{t \to 0+} f(t\mathbf{v}_2) = -1/2$. Having different directional limits along different lines implies f does **not** have a limit at the origin. (See Question 7)

Graph of $xy/(x^2 + y^2)$.



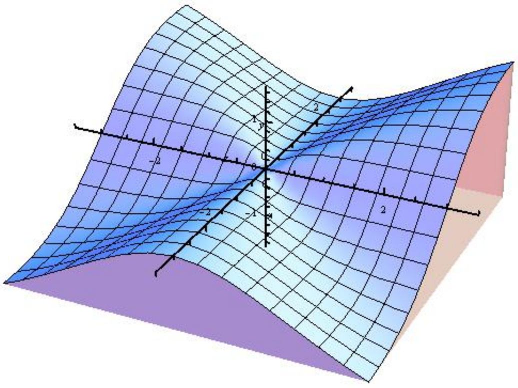
The red line shows the path to the origin along the x - y line, and the blue line the x - $(-y)$ line. There is no value that could be assigned to $f(\mathbf{0})$ which would reconcile these two different limits at 0.

iii. If $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$ then $|\mathbf{x}|^2 = x^2 + y^2$ and $|x|, |y| \leq |\mathbf{x}|$. Therefore, for $\mathbf{x} \neq \mathbf{0}$,

$$|f(\mathbf{x})| \leq \frac{|\mathbf{x}|^3}{|\mathbf{x}|^2} = |\mathbf{x}| \to 0$$

as $\mathbf{x} \to \mathbf{0}$. Thus, by the Sandwich Rule, $f(\mathbf{x}) \to 0$, i.e. the limit exists and is 0.

Graph of $x^2y/(x^2+y^2)$.



iv. Let

$$f(\mathbf{x}) = \frac{xy^2}{x^2+y^4}$$

for $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{x} \neq \mathbf{0}$. Then $f(t\mathbf{e}_1) = 0$ for all $t \neq 0$, so $\lim_{t\to 0+} f(t\mathbf{e}_1) = 0$.

Yet,

$$f\left(\begin{pmatrix} t^2 \\ t \end{pmatrix}\right) = \frac{t^4}{t^4+t^4} = \frac{1}{2},$$

for all $t \neq 0$. Thus $\lim_{t\to 0+} f\left((t^2,t)^T\right) = 1/2$.

Different limits along different curves to the origin means that f does not have a limit at the origin. (See Question 9.)

v. With the intention of using the Sandwich Rule bound the function from above by a simpler function. We do this by bounding the denominator from *below* by a simpler function, i.e. $x^4+y^2 \geq y^2$, **useful** only if $y \neq 0$. For then

$$\left|\frac{xy^2}{x^4+y^2}\right| \leq \left|\frac{xy^2}{y^2}\right| = |x| \leq |\mathbf{x}|.$$

where $\mathbf{x} = (x,y)^T \in \mathbb{R}^2$.

If $y = 0$ then $f(\mathbf{x}) = 0$ which is certainly less than $|\mathbf{x}|$.

So in all cases $|f(\mathbf{x})| \leq |\mathbf{x}|$. Thus, by the Sandwich Rule, $f(\mathbf{x}) \to 0$, as $\mathbf{x} \to \mathbf{0}$, i.e. the limit exists with value 0.

vi. Let

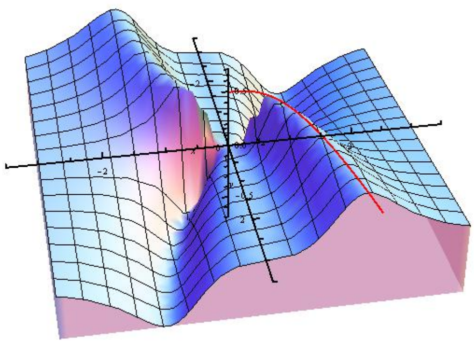
$$f(\mathbf{x}) = \frac{xy^3}{x^2+y^6}$$

for $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{x} \neq \mathbf{0}$. Then the two directional limits

$$\lim_{t \to 0+} f(t\mathbf{e}_1) = 0 \quad \text{and} \quad \lim_{t \to 0+} f\left(\begin{pmatrix} t^3 \\ t \end{pmatrix}\right) = \frac{1}{2},$$

are different. Hence f has no limit at the origin.

The graph of this is too complicated for Mathematica to plot well near the origin, but you should see where the function is 0 on the x-axis and 1/2 on the parabola $y = x^2$ (the red line):



For an example of the limit of a *vector*-valued function we have

12. Find the limit, if it exists, of

$$\lim_{(x,y) \to (0,0)} \left(x^2y + 1, \frac{(xy)^2}{(xy)^2 + (x-y)^2}\right)^T.$$

Solution Let $\mathbf{f} : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}^2$ be given by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x^2y + 1 \\ \dfrac{(xy)^2}{(xy)^2 + (x-y)^2} \end{pmatrix}.$$

The *vector*-valued function has a limit at a point if, and only if, each *component* function has a limit at the point. So examine each component function in turn.

- The first function $(x, y)^T \mapsto x^2y + 1$ has the limit 1 at the origin.

- The second function

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{(xy)^2}{(xy)^2 + (x-y)^2},$$

has no limit at the origin. To see this, if $(x,y)^T = t\mathbf{e}_1 = (t,0)^T$ then the directional limit is 0 as $t \to 0+$. If $(x,y)^T = (t,t)^T$ then the directional limit is 1. Different directional limits imply no limit exists.

Since there is a component function with no limit at the origin the original vector-valued function does **not** have a limit at the origin.

Solutions to Additional Questions 1

13. By verifying the ε - δ definition show that the scalar-valued $g : \mathbb{R}^3 \to \mathbb{R}$, $(x, y)^T \mapsto x^2 y$ has limit 12 at $\mathbf{a} = (2, 3)^T$.

Hint Prove that

$$x^2y - 12 = (x-2)^2 (y-3) + 3(x-2)^2 + 4(x-2)(y-3) + 12(x-2) + 4(y-3).$$

Deduce that if $|x-2|$, $|y-3| < \delta$ and $\delta \le 1$ then

$$\left|x^2y - 12\right| < 24\delta.$$

Solution Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/24)$ and assume $\mathbf{x} = (x, y)^T$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$. Then

$$|x-2| < \delta \text{ and } |y-3| < \delta. \quad (8)$$

To get the result in the hint you can multiply out the right hand side but, more constructively, ask yourself how was it derived? Perhaps by replacing x by $x-2$ and y by $y-3$ in the left hand side. For example,

$$\begin{aligned} x^2y - 12 &= (x-2)^2 (y-3) + 3x^2 + 4xy - 12x - 4y \\ &= (x-2)^2 (y-3) + 3(x-2)^2 + 4(x-2)(y-3) + 12x + 4y - 36 \\ &= (x-2)^2 (y-3) + 3(x-2)^2 + 4(x-2)(y-3) + 12(x-2) + 4(y-3) \end{aligned}$$

Thus

$$\left|x^2y - 12\right| < \delta^3 + 3\delta^2 + 4\delta^2 + 12\delta + 4\delta,$$

by (8). We are also assuming that $\delta \le 1$ in which case $\delta^2 \le \delta$ and $\delta^3 \le \delta$. Thus

$$\left|x^2y - 12\right| < (1 + 3 + 4 + 12 + 4)\,\delta = 24\delta.$$

And, since $\delta \le \varepsilon/24$,

$$\left|x^2y - 12\right| < 24(\varepsilon/24) = \varepsilon.$$

Therefore we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} x^2y = 12$.

14 Verify that the vector-valued function

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x^2y \end{pmatrix}$$

has limit $(5,12)^T$ at $\mathbf{a} = (2,3)^T$.

Note You are not required to verify the definition.

Solution Since we are **not** asked to verify the definition we can use the result that a vector-valued function has a vector limit if, and only if, each scalar-valued component function has the limit of the corresponding component of the limit vector. So we need only show that

$$\lim_{\mathbf{x}\to\mathbf{a}} (x+y) = 5 \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{a}} x^2 y = 12.$$

Yet the first of these limits was the subject of Question 1, the second of Question 13.

15. *In the lectures we need to use the Cauchy-Schwarz inequality* $|\mathbf{a}\bullet\mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$ *and the triangle inequality* $|\mathbf{c}+\mathbf{d}| \leq |\mathbf{c}| + |\mathbf{d}|$, *for vectors* $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \mathbb{R}^n$. *This question is a recap of proofs of these results which you should already know.*

i. Prove that if $a,b,c \in \mathbb{R}$, $a > 0$ and $ax^2 + 2bx + c \geq 0$ for all $x \in \mathbb{R}$ then $b^2 \leq ac$. When do we have equality?

ii. Starting from the true statement that

$$0 \leq \sum_{i=1}^{n} (a_i + b_i x)^2$$

for all $x \in \mathbb{R}$, deduce the Cauchy-Schwarz inequality $|\mathbf{a}\bullet\mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$, written in the form

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

When do we have equality?

Hint *Make use of Part i.*

iii. *Triangle inequality.* Prove that if $\mathbf{c},\mathbf{d} \in \mathbb{R}^n$ then $|\mathbf{c}+\mathbf{d}| \leq |\mathbf{c}| + |\mathbf{d}|$.

Hint: *make use of part iii.*

iv. Prove that if $\mathbf{c},\mathbf{d} \in \mathbb{R}^n$ then $|\mathbf{c}-\mathbf{d}| \geq ||\mathbf{c}| - |\mathbf{d}||$.

Solution i. Complete the square as

$$ax^2 + 2bx + c = a\left(x^2 + \frac{2b}{a}x + \frac{c}{a}\right) = a\left(\left(x + \frac{b}{a}\right)^2 - \frac{b^2}{a^2} + \frac{c}{a}\right), \tag{9}$$

allowable since $a \neq 0$. We are told this is non-negative for **all** x, including $x = -b/a$. Thus

$$a\left(-\frac{b^2}{a^2} + \frac{c}{a}\right) \geq 0. \tag{10}$$

Since $a > 0$ we have $b^2 \leq ac$ as required.

Claim We have $b^2 = ac$ if, and only if, there exists a solution to $ax^2+2bx+c = 0$.

Proof ($\Longrightarrow$) Assume $b^2 = ac$. Then (9) gives

$$ax^2 + 2bx + c = a\left(x + \frac{b}{a}\right)^2$$

which has a zero at $x = -b/a$.

($\Longleftarrow$) Assume there exists $x_0 : ax_0^2 + 2bx_0 + c = 0$. Then by (9) and (10),

$$0 = \left(x_0 + \frac{b}{a}\right)^2 - \frac{b^2}{a^2} + \frac{c}{a} \geq \left(x_0 + \frac{b}{a}\right)^2.$$

This can hold only if $x_0 = -b/a$. Then $ax_0^2 + 2bx_0 + c = 0$ rearranges to $b^2 = ac$. ■

ii. Expand the given inequality, so

$$0 \leq \sum_{i=1}^{n} (a_i + b_i x)^2 = \left(\sum_{i=1}^{n} b_i^2\right) x^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right) x + \left(\sum_{i=1}^{n} a_i^2\right). \tag{11}$$

Thus we can apply Part i with

$$a = \sum_{i=1}^{n} b_i^2, \quad b = \sum_{i=1}^{n} a_i b_i \quad \text{and} \quad c = \sum_{i=1}^{n} a_i^2.$$

The conclusion $b^2 \leq ac$ of Part i. then becomes

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{12}$$

We can rewrite this in terms of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then

$$\begin{aligned} |\mathbf{a} \bullet \mathbf{b}|^2 &= \left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right) \quad \text{by } (12) \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 . \end{aligned}$$

Take positive square roots to get the required inequality.

By Part i. there is equality in (12) iff there is a solution, x_0 say, for (11). Yet the quadratic in (11) is a sum of squares and a sum of squares of real numbers can only be zero if each square is zero. That is $a_i + b_i x_0 = 0$ for all $1 \leq i \leq n$. So we have equality in (12) iff there exists $\lambda = -x_0 \in \mathbb{R} : a_i = \lambda b_i$ for all $1 \leq i \leq n$, i.e. $\mathbf{a} = \lambda \mathbf{b}$.

iii. If $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ then

$$\begin{aligned} |\mathbf{c} + \mathbf{d}|^2 &= (\mathbf{c} + \mathbf{d}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{c} \bullet \mathbf{c} + \mathbf{c} \bullet \mathbf{d} + \mathbf{d} \bullet \mathbf{c} + \mathbf{d} \bullet \mathbf{d} \\ &= |\mathbf{c}|^2 + 2\mathbf{c} \bullet \mathbf{d} + |\mathbf{d}|^2 \\ &\leq |\mathbf{c}|^2 + 2 |\mathbf{c} \bullet \mathbf{d}| + |\mathbf{d}|^2 \\ &\leq |\mathbf{c}|^2 + 2 |\mathbf{c}| |\mathbf{d}| + |\mathbf{d}|^2 \quad \text{by part ii,} \\ &= (|\mathbf{c}| + |\mathbf{d}|)^2 . \end{aligned}$$

Take positive square roots to get the required inequality.

iv. If $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ then

$$|\mathbf{c}| = |\mathbf{c} - \mathbf{d} + \mathbf{d}| \leq |\mathbf{c} - \mathbf{d}| + |\mathbf{d}|$$

by part iii. Simply rearrange as $|\mathbf{c} - \mathbf{d}| \geq |\mathbf{c}| - |\mathbf{d}|$. Yet, by swapping $\mathbf{c}$ and $\mathbf{d}$, $|\mathbf{c} - \mathbf{d}| = |\mathbf{d} - \mathbf{c}| \geq |\mathbf{d}| - |\mathbf{c}|$, by result just proved. Thus

$$|\mathbf{c} - \mathbf{d}| \geq |\mathbf{c}| - |\mathbf{d}| \text{ and } |\mathbf{d}| - |\mathbf{c}| ,$$

which can be summed up in the one inequality $|\mathbf{c} - \mathbf{d}| \geq ||\mathbf{c}| - |\mathbf{d}||$.